

# FREQUENCY SHIFT UP TO THE 2-PM APPROXIMATION

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**Abstract.** A lot of fundamental tests of gravitational theories rely on highly precise measurements of the travel time and/or the frequency shift of electromagnetic signals propagating through the gravitational field of the Solar System. In practically all of the previous studies, the explicit expressions of such travel times and frequency shifts as predicted by various metric theories of gravity are derived from an integration of the null geodesic differential equations. However, the solution of the geodesic equations requires heavy calculations when one has to take into account the presence of mass multipoles in the gravitational field or the tidal effects due to the planetary motions, and the calculations become quite complicated in the post-post-Minkowskian approximation. This difficult task can be avoided using the time transfer function's formalism. We present here our last advances in the formulation of the one-way frequency shift using this formalism up to the post-post-Minkowskian approximation.

Keywords: frequency shift, relativity, fundamental physics, space navigation

## 1 Introduction

The treatment of light propagation in a relativistic framework is extremely important for various fields of study such as fundamental physics and astronomy, astrophysics and space navigation. Attaining very accurate measurements could allow us to observe a new range of subtle physical effects. Nowadays, a few approaches exist to model light propagation in a relativistic context. Among them, the post-Newtonian (pN) and the post-Minkowskian (pM) approximations (see for example Kopeikin & Schäfer 1999; Klioner & Zschocke 2010) are those mainly used in order to find perturbative solutions of the null geodesic equation.

In this work, an alternative formulation to compute the one way frequency shift of an electromagnetic signal is presented. Being based on the time transfer function formalism (Teyssandier & Le Poncin-Lafitte 2008), it allows us to compute this observable up to the post-post-Minkowskian approximation without integrating the null geodesic equation, allowing lighter calculations.

Section 2 contains the notations and conventions used in this document. In section 3 we give our framework and the definition of the one way frequency shift while in section 4 we provide the relations between the frequency shift and the time transfer function. These quantities will be then used in section 5 to obtain the post-Minkowskian expansion of the observable up to the 2PM approximation. Our conclusions and possible applications of this study are given in section 6.

## 2 Notation and conventions

In this paper  $c$  is the speed of light in a vacuum and  $G$  is the Newtonian gravitational constant. The Lorentzian metric of space-time  $V_4$  is denoted by  $g$ . The signature adopted for  $g$  is  $(+ - - -)$ . We suppose that space-time is covered by some global quasi-Galilean coordinate system  $(x^\mu) = (x^0, \mathbf{x})$ , where  $x^0 = ct$ ,  $t$  being a time coordinate, and  $\mathbf{x} = (x^i)$ . We assume that the curves of equations  $x^i = \text{const}$  are timelike, which means that  $g_{00} > 0$  anywhere. We employ the vector notation  $\mathbf{a}$  in order to denote  $(a^1, a^2, a^3) = (a^i)$ . For any quantity  $f(x^\lambda)$ ,  $f_{,\alpha}$  denotes the partial derivative of  $f$  with respect to  $x^\alpha$ . The indices in parentheses characterize the order of perturbation. They are set up or down, depending on the convenience.

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### 3 The one-way frequency shift

Consider a clock  $\mathcal{O}_A$  located at point  $\mathcal{A}$  and a clock  $\mathcal{O}_B$  located at point  $\mathcal{B}$  delivering, respectively, the proper frequency  $\nu_A$  and  $\nu_B$ . Then, suppose that  $\mathcal{O}_A$  is sending an electromagnetic signal to  $\mathcal{O}_B$  along null geodesics of the metric (geometric optics approximation). Then, the one way frequency shift is defined by

$$\left. \frac{\Delta\nu}{\nu} \right|_{A \rightarrow B}^{\text{one-way}} = \frac{\nu_B}{\nu_A} - 1. \quad (3.1)$$

It is well-known that the ratio  $\nu_B/\nu_A$  can be expressed as (Synge 1960)

$$\frac{\nu_B}{\nu_A} = \frac{u_B^\mu k_\mu^B}{u_A^\nu k_\nu^A} = \frac{k_0^B u_B^0 + u_B^i \hat{k}_i^B}{k_0^A u_A^0 + u_A^i \hat{k}_i^A} = \left( \frac{d\tau}{dt} \right)_A \frac{dt_A}{dt_B} \left( \frac{dt}{d\tau} \right)_B, \quad (3.2)$$

where  $u_{A/B}^\mu = (dx^\mu/ds)_{A/B}$  are the four-velocity of observers  $\mathcal{A}$  and  $\mathcal{B}$ ,  $\hat{k}_i = \left( \frac{k_i}{k_0} \right)$  and  $k_\mu^A$  and  $k_\mu^B$  are the wave vectors (the null tangent vectors) at the point of emission  $x_A$  and at the point of reception  $x_B$ , respectively. Terms appearing in the right hand side of Eq. (3.2) can be expressed as

$$\left( \frac{d\tau}{dt} \right)_{A/B} = [g_{00} + 2g_{0i}\beta^i + g_{ij}\beta^i\beta^j]_{A/B}^{1/2}, \quad \frac{dt_A}{dt_B} = \frac{k_0^B}{k_0^A} \frac{1 + \beta_B^i \hat{k}_i^B}{1 + \beta_A^i \hat{k}_i^A}, \quad (3.3)$$

with  $\beta_{A/B}^i = \frac{1}{c} \frac{dx_{A/B}^i}{dt}$  being the coordinate velocities of observers  $\mathcal{A}$  and  $\mathcal{B}$ .

### 4 Relation between frequency shift and time transfer functions

We put  $x_A = (ct_A, \mathbf{x}_A)$  the event of emission  $\mathcal{A}$  and  $x_B = (ct_B, \mathbf{x}_B)$  the event of reception  $\mathcal{B}$ . Moreover, we define  $\mathcal{T}_e$  and  $\mathcal{T}_r$  as two distinct (coordinate) time transfer functions defined as

$$t_B - t_A = \mathcal{T}_e(t_A, \mathbf{x}_A, \mathbf{x}_B) = \mathcal{T}_r(t_B, \mathbf{x}_A, \mathbf{x}_B). \quad (4.1)$$

The relations between time transfer functions and the wave vectors  $k^\mu = dx^\mu/d\lambda$  at emission and reception has been derived by Le Poncin-Lafitte et al. (2004) :

$$\left( \hat{k}_i \right)_A = \left( \frac{k_i}{k_0} \right)_A = c \frac{\partial \mathcal{T}_e}{\partial x_A^i} \left[ 1 + \frac{\partial \mathcal{T}_e}{\partial t_A} \right]^{-1} = c \frac{\partial \mathcal{T}_r}{\partial x_A^i}, \quad (4.2a)$$

$$\left( \hat{k}_i \right)_B = \left( \frac{k_i}{k_0} \right)_B = -c \frac{\partial \mathcal{T}_e}{\partial x_B^i} = -c \frac{\partial \mathcal{T}_r}{\partial x_B^i} \left[ 1 - \frac{\partial \mathcal{T}_r}{\partial t_B} \right]^{-1}, \quad (4.2b)$$

$$\frac{(k_0)_B}{(k_0)_A} = \left[ 1 + \frac{\partial \mathcal{T}_e}{\partial t_A} \right]^{-1} = 1 - \frac{\partial \mathcal{T}_r}{\partial t_B}, \quad (4.2c)$$

where  $\mathcal{T}_e$  and  $\mathcal{T}_r$  are evaluated at the event of emission  $\mathcal{A}$  and at the event of reception  $\mathcal{B}$  respectively. It's then straightforward to define the one-way frequency shift (3.1) as a function of  $\mathcal{T}_{e/r}$  and their partial derivatives.

### 5 Post-Minkowskian expansion of the frequency shift

The expression of the time transfer functions  $\mathcal{T}_{e/r}$  as a formal post-Minkowskian series has been derived by Teyssandier & Le Poncin-Lafitte (2008). In this communication, we focus on  $\mathcal{T}_r$ , but similar considerations hold for  $\mathcal{T}_e$ .  $\mathcal{T}_r$  can be written in ascending power of  $G$  defined as

$$\mathcal{T}_r(\mathbf{x}_A, t_B, \mathbf{x}_B) = \frac{R_{AB}}{c} + \frac{1}{c} \sum_{n=1}^{\infty} \Delta_r^{(n)}(\mathbf{x}_A, t_B, \mathbf{x}_B), \quad (5.1)$$

where  $\Delta_r^{(n)}$  is of the order  $\mathcal{O}(G^n)$ ,  $R_{AB}^i = x_B^i - x_A^i$ ,  $R_{AB} = |R_{AB}^i|$  and  $N^i = \frac{R_{AB}^i}{R_{AB}}$ .

Then, we can express the one-way frequency shift (3.2) as follows (in agreement with the expression found by Hees et al. (2012))

$$\frac{\nu_B}{\nu_A} = \frac{[g_{00} + 2g_{0i}\beta^i + g_{ij}\beta^i\beta^j]_A^{1/2}}{[g_{00} + 2g_{0i}\beta^i + g_{ij}\beta^i\beta^j]_B^{1/2}} \times \frac{1 - N^i\beta_B^i - \beta_B^i \frac{\partial \Delta_r}{\partial x_B^i} - \frac{\partial \Delta_r}{\partial t_B}}{1 - N^i\beta_A^i + \beta_A^i \frac{\partial \Delta_r}{\partial x_A^i}}. \quad (5.2)$$

The goal of this work is to provide a new way of computing a general form for the derivatives of the time delay function up to the post-post Minkowskian order. In order to do so, we rewrite  $\Delta_r^{(1)}$  and  $\Delta_r^{(2)}$  given by Teyssandier & Le Poncin-Lafitte (2008) as

$$\Delta_r^{(1)}(\mathbf{x}_A, t_B, \mathbf{x}_B) = \frac{R_{AB}}{2} \int_0^1 \left[ g_{(1)}^{00} - 2N^i g_{(1)}^{0i} + N^i N^j g_{(1)}^{ij} \right]_{z^\alpha(\lambda)} d\lambda = \int_0^1 m_{(1)}(\lambda) d\lambda, \quad (5.3a)$$

$$\Delta_r^{(2)}(\mathbf{x}_A, t_B, \mathbf{x}_B) = \int_0^1 [\mathcal{I}_1(\lambda) + \mathcal{I}_2(\lambda) + \mathcal{I}_3(\lambda)] d\lambda, \quad (5.3b)$$

where

$$\mathcal{I}_1 = m_{(2)}(\lambda) - \Delta_r^{(1)}(\mathbf{z}(\lambda), t_B, \mathbf{x}_B) m_{(1),0}(\lambda), \quad (5.4a)$$

$$\mathcal{I}_2 = \left[ R_{AB} g_{(1)}^{0i} - R_{AB}^k g_{(1)}^{ik} \right]_{z^\alpha(\lambda)} \frac{\partial \Delta_r^{(1)}}{\partial x^i}(\mathbf{z}(\lambda)), \quad (5.4b)$$

$$\mathcal{I}_3 = -\frac{R_{AB}}{2} \sum_{j=1}^3 \left[ \frac{\partial \Delta_r^{(1)}}{\partial x^j}(\mathbf{z}(\lambda)) \right]^2, \quad (5.4c)$$

$\mathbf{z}(\lambda) = x_B^i - \lambda R_{AB}^i$  being the spatial composantes of the Minkowskian straight line of equation  $z^\alpha(\lambda) = (x_B^0 - \lambda R_{AB}, \mathbf{z}(\lambda))$  and

$$m_{(n),\alpha}(\lambda) = \frac{R_{AB}}{2} \left[ g_{(n),\alpha}^{00} - 2N^i g_{(n),\alpha}^{0i} + N^i N^j g_{(n),\alpha}^{ij} \right]_{z^\beta(\lambda)}, \quad (5.5a)$$

$$\frac{\partial \Delta_r^{(1)}}{\partial x^i}(\mathbf{z}(\lambda)) = \int_0^1 \left[ m_{(1),\alpha}(\lambda\mu) z_{A,i}^\alpha(\lambda\mu) + \tilde{h}_{(1)}^i(\lambda\mu) \right] d\mu,$$

with  $z_{A/B,i}^\alpha = \partial z^\alpha / \partial x_{A/B}^i$ . The derivatives of the first PM order of the delay function can be then easily calculated

$$\frac{\partial \Delta_r^{(1)}}{\partial x_{A/B}^i}(\mathbf{x}_A, t_B, \mathbf{x}_B) = \int_0^1 \left[ m_{(1),\alpha}(\lambda) z_{A/B,i}^\alpha(\lambda) \pm \tilde{h}_{(1)}^i(\lambda) \right] d\lambda, \quad (5.6a)$$

$$\frac{\partial \Delta_r^{(1)}}{\partial t_B}(\mathbf{x}_A, t_B, \mathbf{x}_B) = \int_0^1 [m_{(1),0}(\lambda) c] d\lambda, \quad (5.6b)$$

where the function  $\tilde{h}$  is defined by

$$\tilde{h}_{(n),j}^i(\lambda) = \frac{\partial m_{(n),j}}{\partial x_A^i} \Big|_{z^\alpha = \text{cst}} = - \frac{\partial m_{(n),j}}{\partial x_B^i} \Big|_{z^\alpha = \text{cst}} = \frac{1}{2} \left[ -N^i g_{(n),j}^{00} + 2g_{(n),j}^{0i} - 2g_{(n),j}^{ik} N^k + N^k N^l N^i g_{(n),j}^{kl} \right]_{z^\alpha(\lambda)}. \quad (5.7)$$

These equations are equivalent to those derived by Hees et al. (2012). The same approach can be used for the 2PM order resulting in more complex formulas

$$\frac{\partial \Delta_r^{(2)}}{\partial x_{A/B}^i}(\mathbf{x}_A, t_B, \mathbf{x}_B) = \int_0^1 \left[ \frac{\partial \mathcal{I}_1}{\partial x_{A/B}^i} + \frac{\partial \mathcal{I}_2}{\partial x_{A/B}^i} + \frac{\partial \mathcal{I}_3}{\partial x_{A/B}^i} \right] d\lambda \quad (5.8a)$$

$$\frac{\partial \Delta_r^{(2)}}{\partial t_B}(\mathbf{x}_A, t_B, \mathbf{x}_B) = \int_0^1 \left[ \frac{\partial \mathcal{I}_1}{\partial t_B} + \frac{\partial \mathcal{I}_2}{\partial t_B} + \frac{\partial \mathcal{I}_3}{\partial t_B} \right] d\lambda \quad (5.8b)$$

where the derivatives can be expressed as follows

$$\frac{\partial \mathcal{I}_1}{\partial x_{A/B}^i} = m_{(2),\alpha} z_{A/B,i}^\alpha \pm \tilde{h}_{(2)}^i - \Delta_r^{(1)}(z(\lambda), t_b, \mathbf{x}_B) \left[ m_{(1),0\alpha} z_{A/B,i}^\alpha \pm \tilde{h}_{(1),0}^i \right] - m_{(1),0} \frac{\partial \Delta_r^{(1)}}{\partial x_{A/B}^i}(z(\lambda)), \quad (5.9a)$$

$$\begin{aligned} \frac{\partial \mathcal{I}_2}{\partial x_{A/B}^i} = & \left[ \mp N^i g_{(1)}^{0j} \pm g_{(1)}^{ij} + (R_{AB} g_{(1),\alpha}^{0j} - g_{(1),\alpha}^{jk} R_{AB}^k) z_{A/B,i}^\alpha \right] \frac{\partial \Delta_r^{(1)}}{\partial x^j}(z(\lambda)) \\ & + [R_{AB} g_{(1)}^{0j} - R_{AB}^k g_{(1)}^{jk}] \frac{\partial^2 \Delta_r^{(1)}}{\partial x_{A/B}^i \partial x^j}(z(\lambda)), \end{aligned} \quad (5.9b)$$

$$\frac{\partial \mathcal{I}_3}{\partial x_{A/B}^i} = \pm \frac{N_{AB}^i}{2} \sum_{j=1}^3 \left( \frac{\partial \Delta_r^{(1)}}{\partial x^j}(z(\lambda)) \right)^2 - R_{AB} \sum_{j=1}^3 \left[ \frac{\partial \Delta_r^{(1)}}{\partial x^j}(z(\lambda)) \cdot \frac{\partial^2 \Delta_r^{(1)}}{\partial x_{A/B}^i \partial x^j}(z(\lambda)) \right], \quad (5.9c)$$

all quantities being taken at  $(\lambda)$  and where we define

$$\frac{\partial^2 \Delta_r^{(1)}}{\partial x_{A/B}^i \partial x^j}(z(\lambda)) = \int_0^1 [m_{(1),\alpha\beta} z_{A,j}^\alpha z_{A/B,i}^\beta \pm \tilde{h}_{(1),\alpha}^i z_{A,j}^\alpha + m_{(1),\alpha} z_{AA/AB,ji}^\alpha + \tilde{h}_{(1),\alpha}^j z_{A/B,i}^\alpha \pm \tilde{h}_{(1)}^{ji}]_{\lambda\mu} d\mu, \quad (5.10)$$

with the function  $\tilde{h}$  defined by

$$\tilde{h}_{(n)}^{ik} = \left. \frac{\partial \tilde{h}_{(n)}^i}{\partial x_A^k} \right|_{z^\alpha = \text{cst}} = - \left. \frac{\partial \tilde{h}_{(n)}^i}{\partial x_B^k} \right|_{z^\alpha = \text{cst}}, \quad (5.11)$$

and  $z_{AA/AB,ji}^\alpha = \frac{\partial^2 z^\alpha}{\partial x_A^j \partial x_{A/B}^i}$ . Similar expressions can be written for  $\partial \mathcal{I}_i / \partial t_B$ .

## 6 Conclusions

We presented here our last advances in the formulation of the one-way frequency shift up to the post-post-Minkowskian approximation. The main result is given by Eq. (5.2) where the derivatives of  $\Delta_r$  are given up to 2PM order by Eqs. (5.6-5.8). The advantage of our formulation is that it does not require the integration of the null geodesic differential equations. Instead, the frequency shift is expressed as integral of functions defined from the metric (and its derivatives) performed over a Minkowskian straight line.

Exact formulas up to 2PM order may be required for future space missions exploring the inner Solar System as shown by Tommei et al. (2010). The formulas presented here are useful to derive the frequency shift directly from the space-time metric. They can be used to derive the frequency shift up to 2PM order in a Schwarzschild space-time to validate results from Tommei et al. (2010). One can also use them to derive the frequency shift in the field of an ensemble of moving point masses in PPN formalism or in a framework improving the current IAU conventions (Minazzoli & Chauvineau 2009). The present results can also be used to derive frequency shift in alternative theories of gravity if the corresponding space-time metric is known. These applications will be presented elsewhere in a forthcoming paper.

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